Reading a Phase Portrait

Derivation notes - damped harmonic oscillator

1. Equation of motion

A mass m on a spring of stiffness k with linear damping λ obeys

$$m\ddot{x} + \lambda\dot{x} + kx = 0.$$

Dividing by m and writing $\omega_0 = \sqrt{k/m}$ and $\zeta = \lambda/(2\sqrt{km})$,

$$\ddot{x} + 2\zeta\omega_0\,\dot{x} + \omega_0^2\,x = 0.$$

2. State-space form

Let $v = \dot{x}$. The second-order equation becomes a first-order system

$$\dot{x} = v, \qquad \dot{v} = -\,\omega_0^2\,x \;-\; 2\zeta\omega_0\,v.$$

Each point (x, v) is a state; the system assigns it a velocity vector $(\dot{x}, \dot{v})$ - the arrows drawn in the phase portrait.

3. Eigenvalues and regimes

Seeking $x \propto e^{st}$ gives the characteristic equation

$$s^2 + 2\zeta\omega_0\,s + \omega_0^2 = 0 \quad\Rightarrow\quad s_\pm = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}\,.$$

- $0 < \zeta < 1$: complex roots $\Rightarrow$ inward spiral (underdamped).
- $\zeta = 1$: repeated real root $\Rightarrow$ critical damping.
- $\zeta > 1$: two negative real roots $\Rightarrow$ straight decay (overdamped).
- $\zeta = 0$: purely imaginary $\Rightarrow$ closed orbits (undamped).

4. Energy view

With total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, differentiating gives

$$\dot{E} = -\,\lambda\,v^2 \;\le\; 0,$$

so damping drains energy monotonically and every trajectory settles to the origin - except the undamped case, where E is conserved and orbits persist.